

# Mathematical Induction

## Part One

Everybody – do the wave!

# The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let  $P$  be some predicate. The *principle of mathematical induction* states that if

If it starts true...

$P(0)$  is true

...and it stays  
true...

and

$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}. P(n)$

...then it's  
always true.

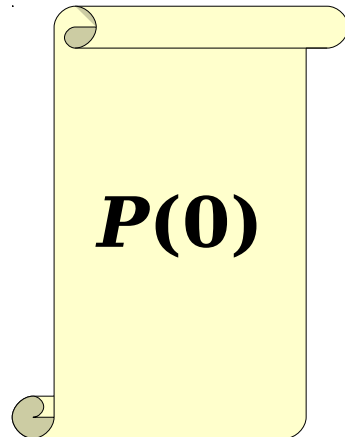
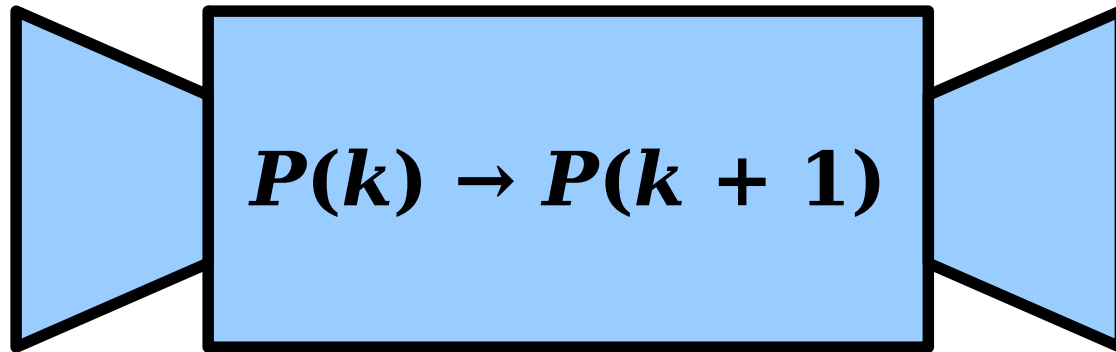
# Induction, Intuitively

$$P(0)$$

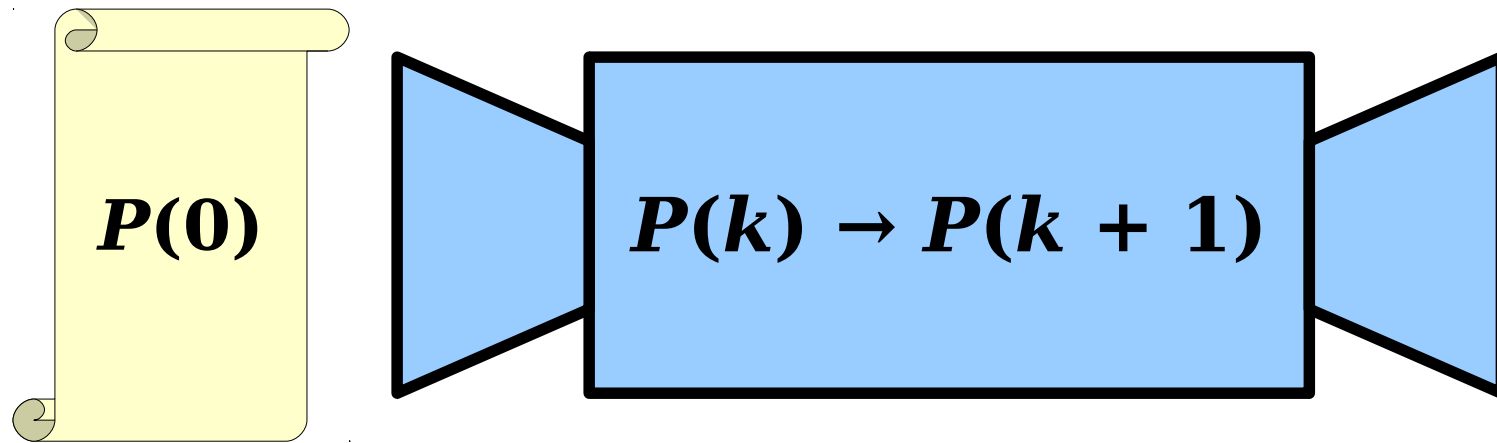
$$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- ...

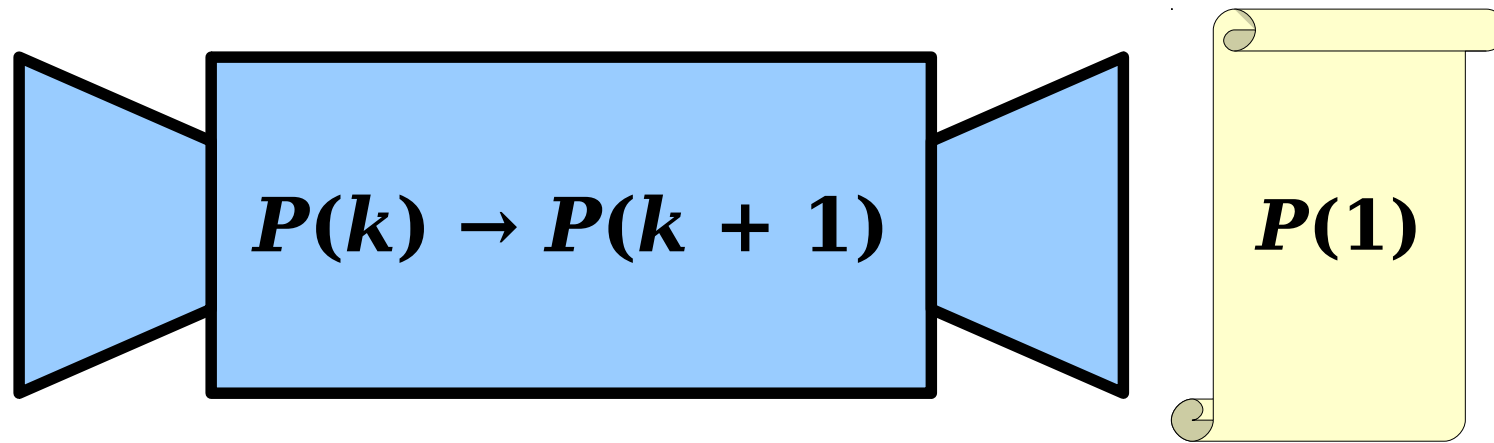
# Why Induction Works



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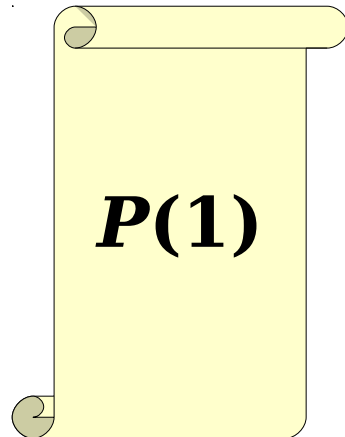
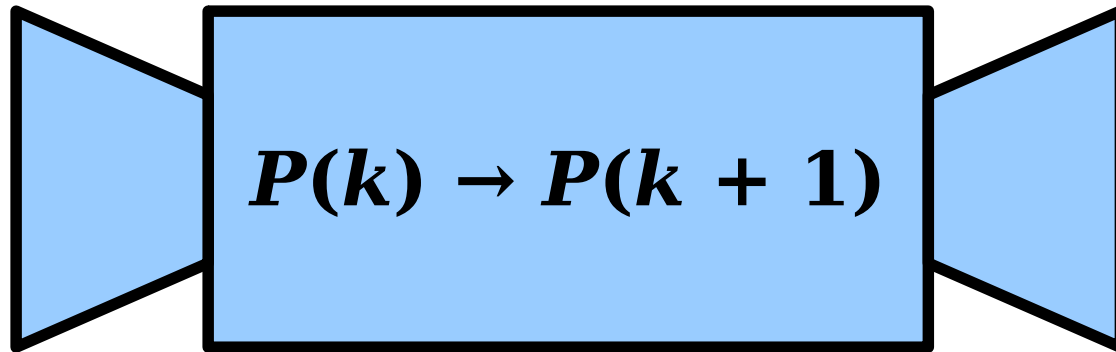


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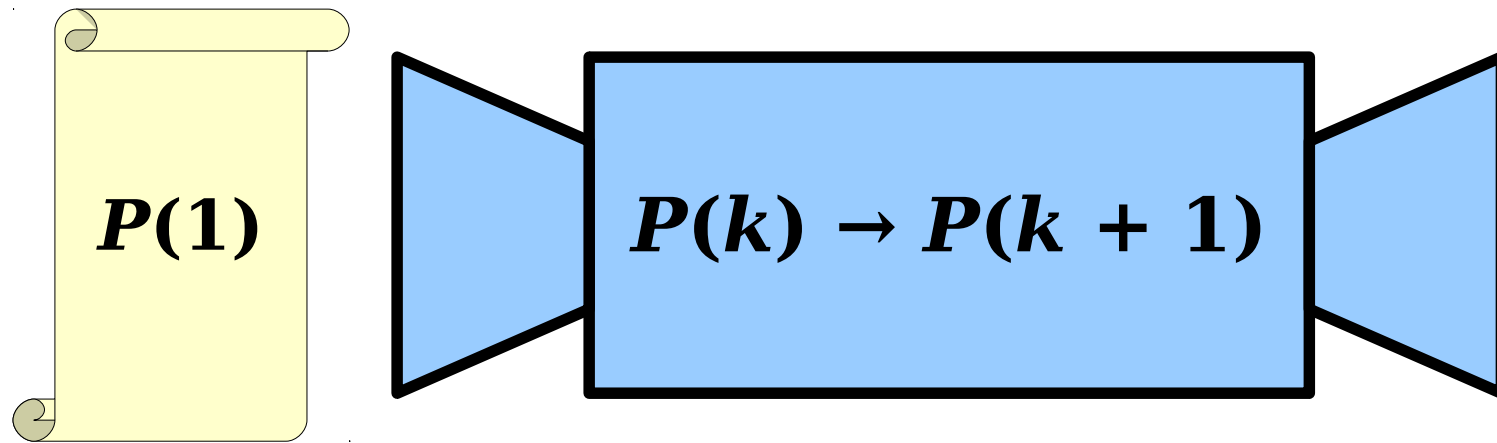




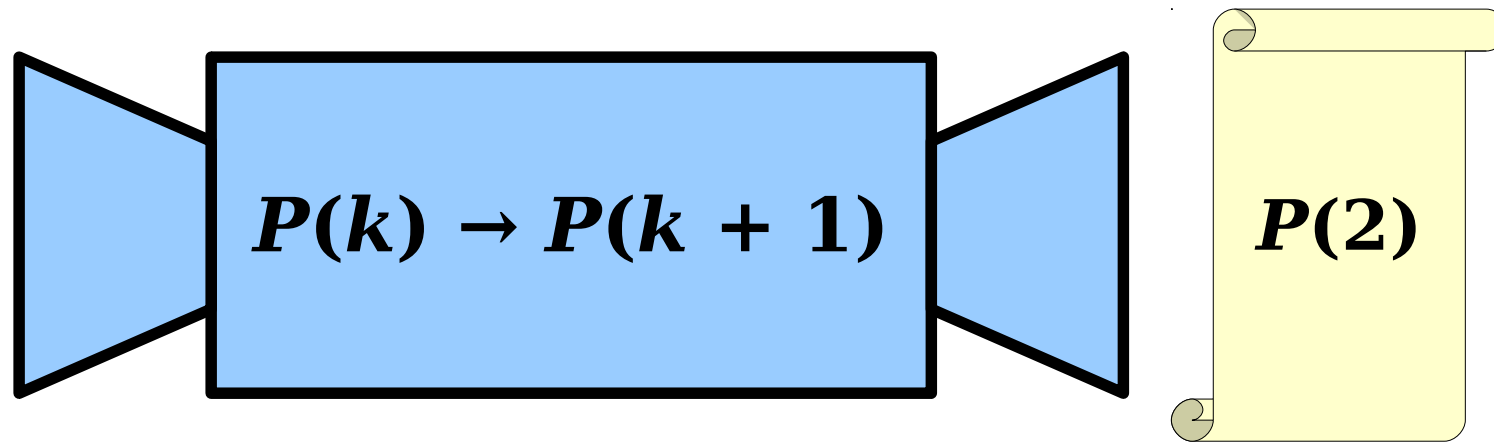
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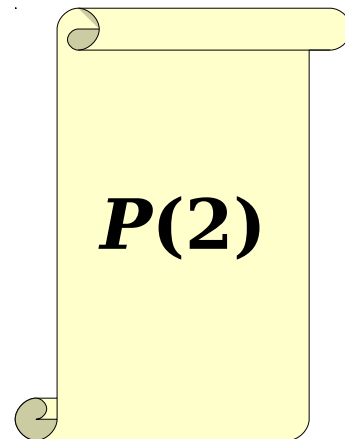
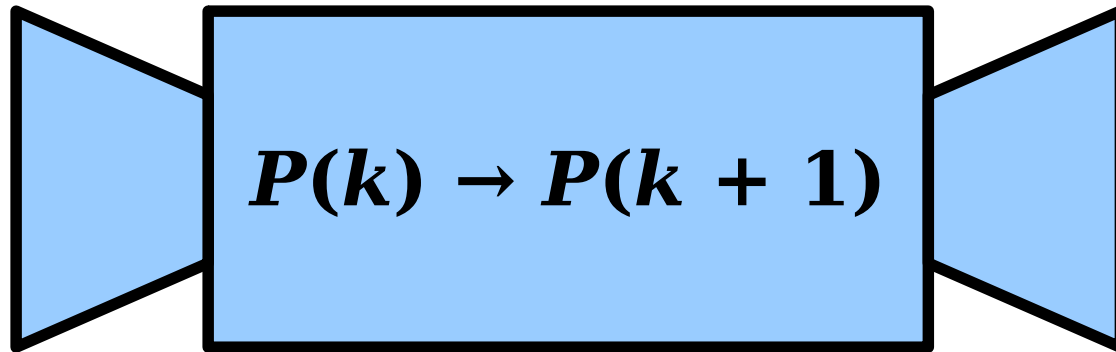
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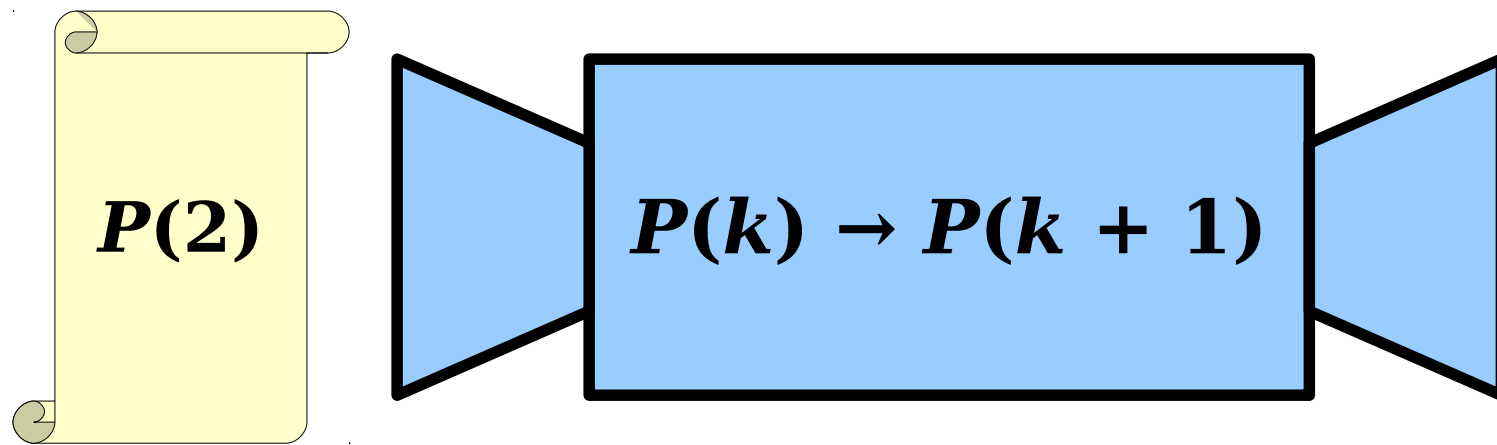
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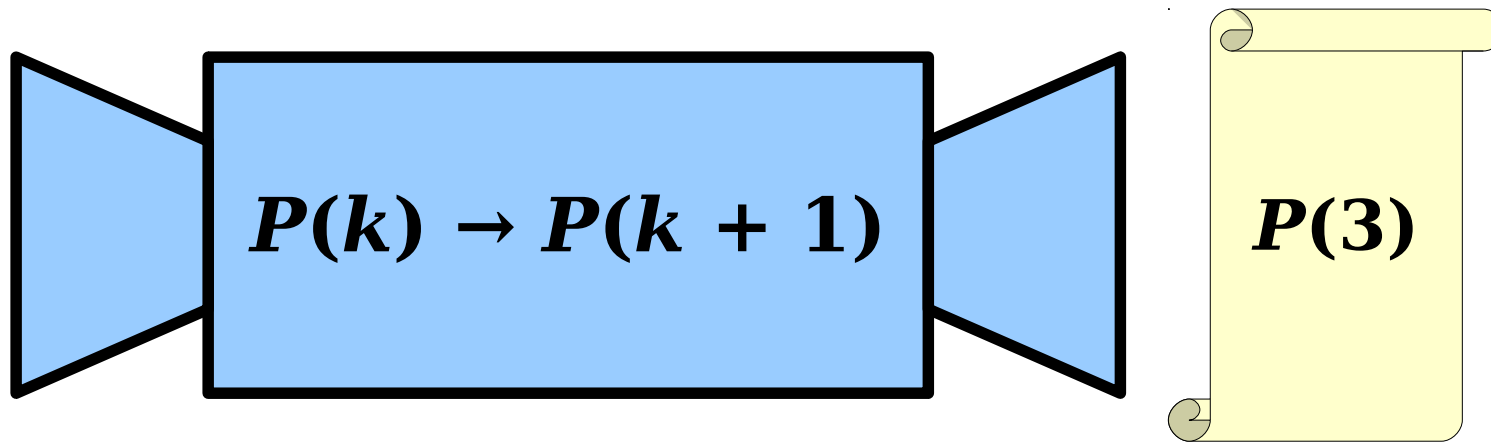
# Why Induction Works



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# Why Induction Works



# Proof by Induction

- A **proof by induction** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers  $n$ .
- In a proof by induction, there are three steps:
  - Prove that  $P(0)$  is true.
    - This is called the **basis step** or the **base case**.
  - Prove that for all naturals  $k$ , if  $P(k)$  is true, then  $P(k+1)$  is true.
    - This is called the **inductive step**. (This part is generally done as a direct proof of a universal implication.)
    - The assumption that  $P(k)$  is true is called the **inductive hypothesis**.
  - Conclude, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ .

Some Sums



$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$2^0 = 1$$

$$2^0 + 2^1 = 1 + 2 = 3$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$$

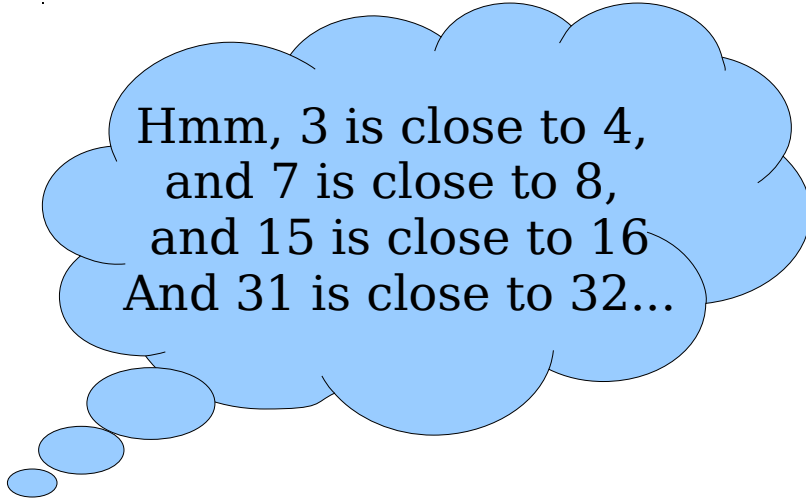
$$2^0 = 1$$

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Hmm, 3 is close to 4,  
and 7 is close to 8,  
and 15 is close to 16  
And 31 is close to 32...

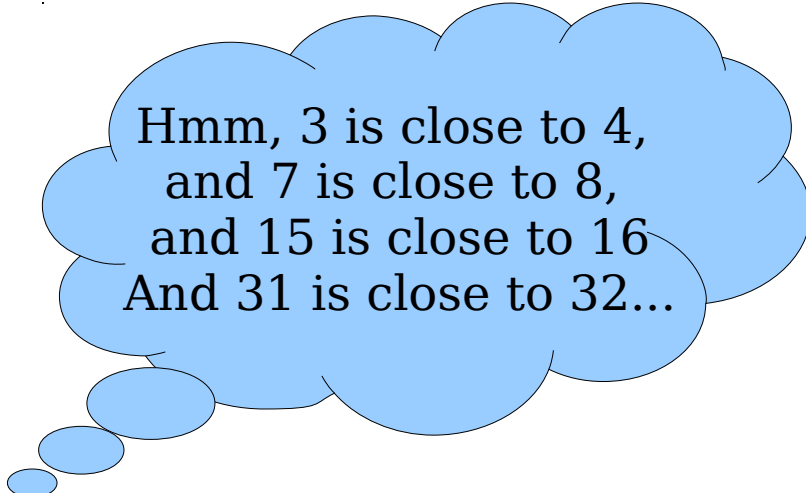
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Hmm, 3 is close to 4,  
and 7 is close to 8,  
and 15 is close to 16  
And 31 is close to 32...

**Quick check:** What is  
the pattern? The sum of  
powers of 2 up to  $2^n$  is...  
Go to  
[PollEv.com/cs103spr25](https://www.pollEv.com/cs103spr25)

$$2^0 = 1 = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

***Theorem:*** The sum of the first  $n$  powers of two is  $2^n - 1$ .

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers  $n$ , then tell them we're going to prove it by induction.

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Here, we state what  $P(0)$  actually says. Now, can go prove this using any proof techniques we'd like!

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For our base case, we need to show  $P(0)$  is true, that the sum of the first zero powers of two is zero as well, we see that  $P(0)$  is true.

**Quick check:** Ignore that this is a new kind of proof for us for a second—if you were proving the theorem in blue by **direct proof**, what would your first sentence be?

Go to [PollEv.com/cs103spr25](https://pollev.com/cs103spr25)

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For the inductive step, pick an arbitrary  $k \in \mathbb{N}$  and assume that  $P(k)$  holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \quad (1)$$

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The goal of this step is to prove

**“For all  $k \in \mathbb{N}$ , if  $P(k)$ , then  $P(k+1)$ .”**

To do this, we'll choose an arbitrary  $k$ , assume that  $P(k)$  is true, then try to prove  $P(k+1)$ .

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Here, we explicitly state  $P(k+1)$ , which is what we want to prove. Now, we can use any proof technique we want to prove it.

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$$2^0 + 2^1 + \dots + 2^{k-1} + 2^k = (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k$$

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Here, we'll use our **inductive hypothesis** (the assumption that  $P(k)$  is true) to simplify a complex expression. This is a common theme in inductive proofs.

For our base case, let  $n = 0$ . The sum of the first 0 powers of two is zero, and  $2^0 - 1 = 0$ . So  $P(0)$  holds.

For the inductive step, pick an arbitrary  $k \in \mathbb{N}$  and assume that  $P(k)$  holds, meaning that

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In a proof by induction, we need to prove that

- ✓  $P(0)$  is true
- ✓ For all  $k \in \mathbb{N}$ , if  $P(k)$  is true, then  $P(k+1)$  is true.

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^k + 2^{k+1} &= (2^0 + 2^1 + \dots + 2^k) + 2^{k+1} \\ &= (2^k - 1) + 2^{k+1} \\ &= 2^k - 1 + 2^{k+1} \\ &= 2^k - 1 + 2 \cdot 2^k \\ &= 2^k - 1 + 2^{k+1} \\ &= 2^{k+1} - 1. \end{aligned}$$

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# A Quick Aside

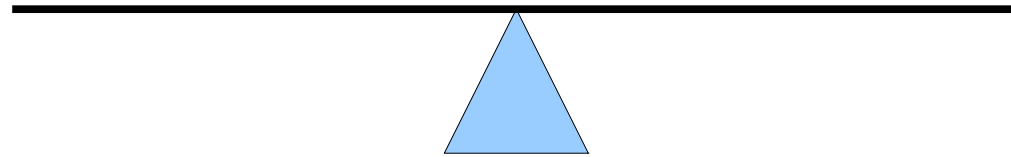
- This result helps explain the range of numbers that can be stored in an **int**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by  $1 + 2 + 4 + 8 + \dots + 2^{31} = 2^{32} - 1$ .
- This formula for sums of powers of two has many other uses as well. You'll see one next time.

# The Counterfeit Coin Problem

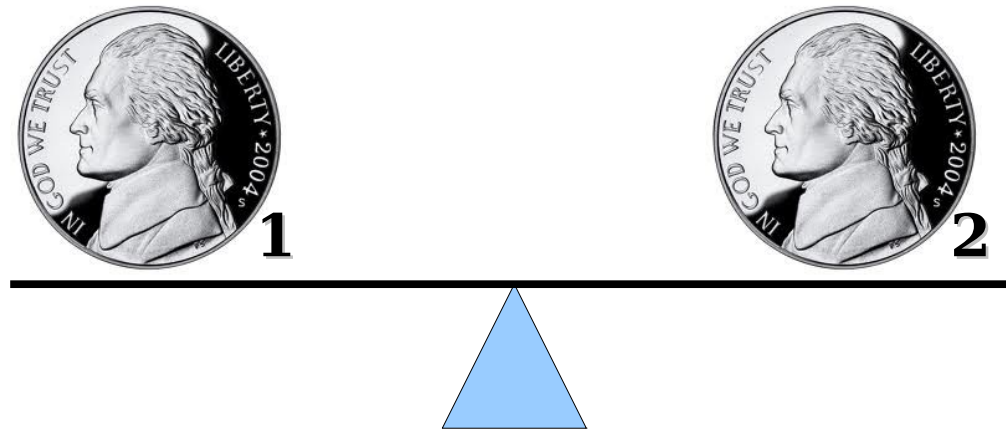
# Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

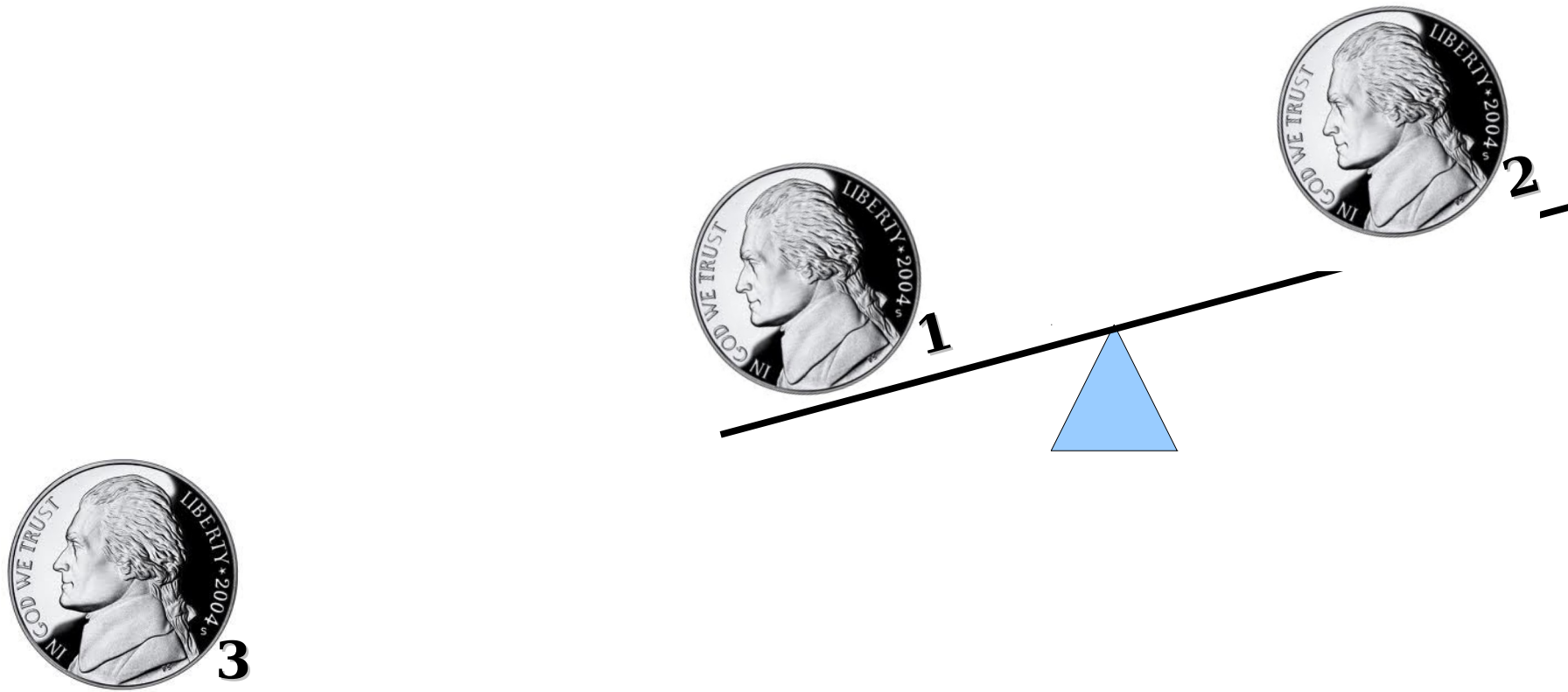
# Finding the Counterfeit Coin



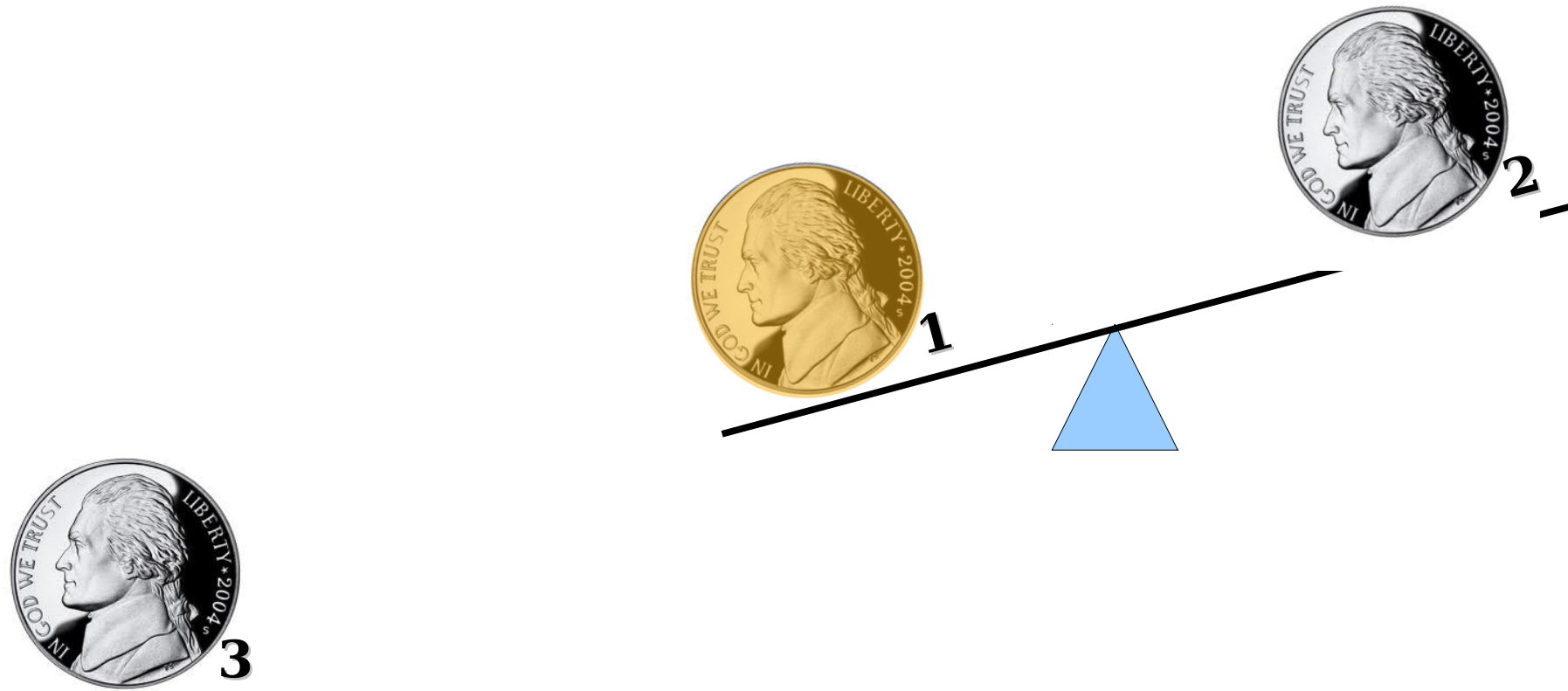
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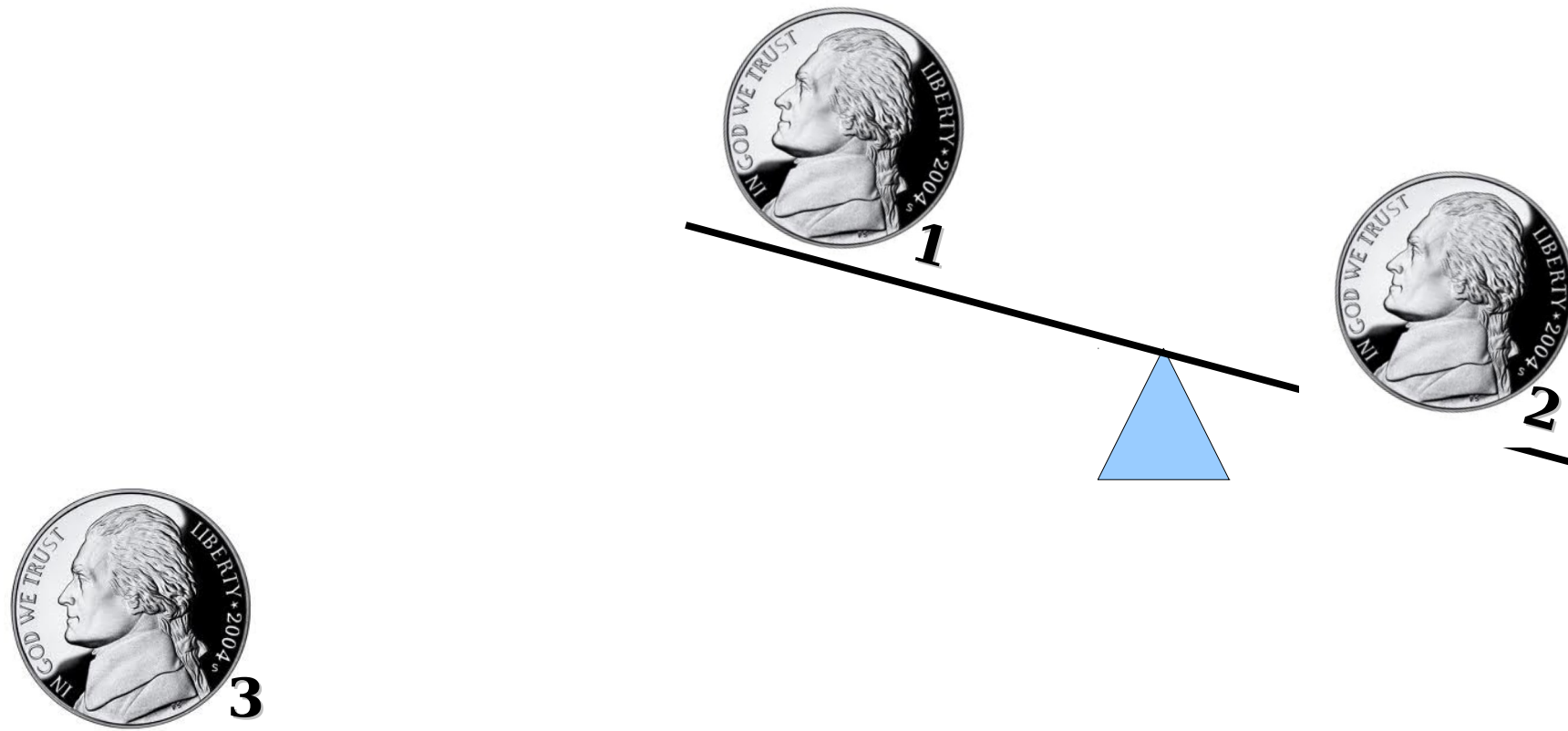


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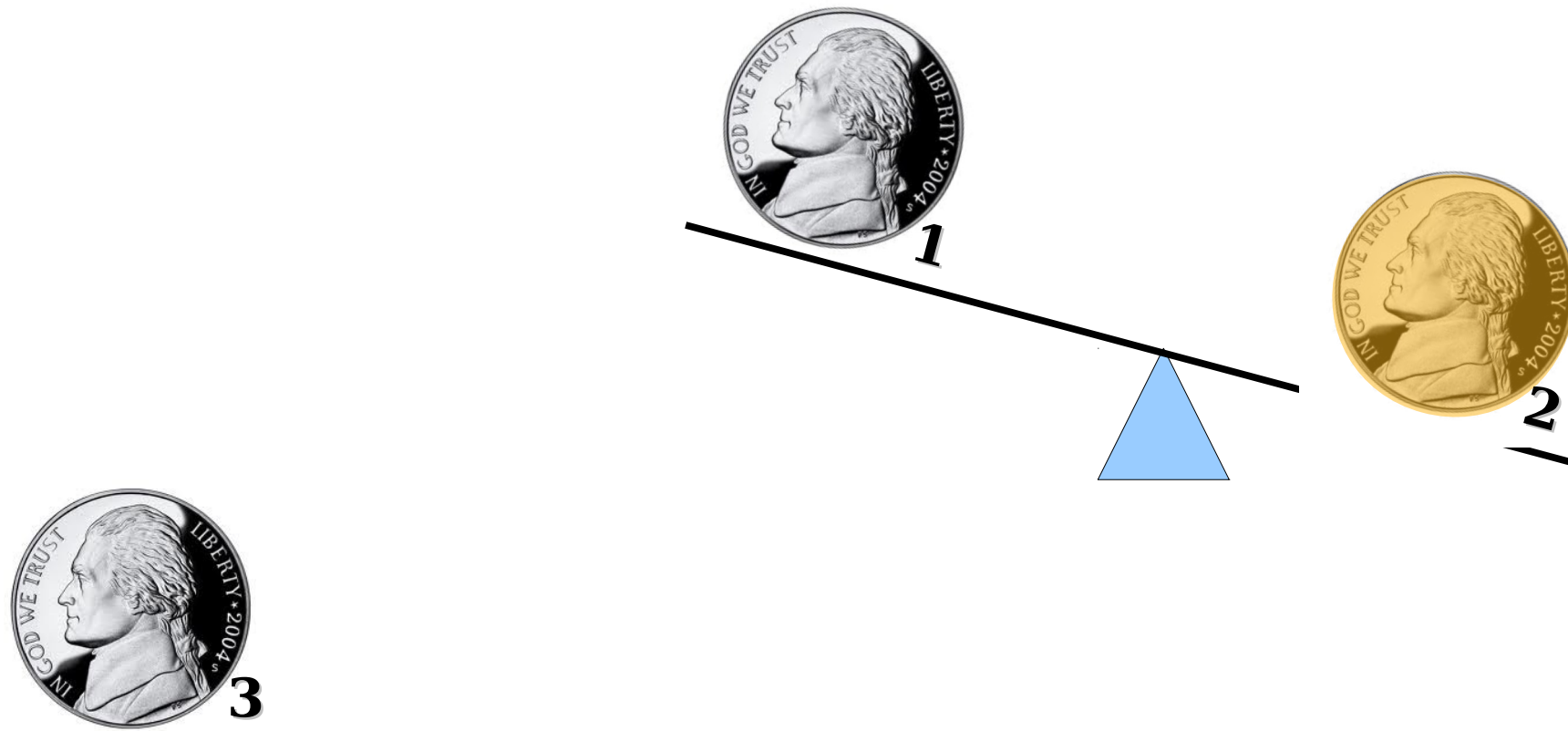




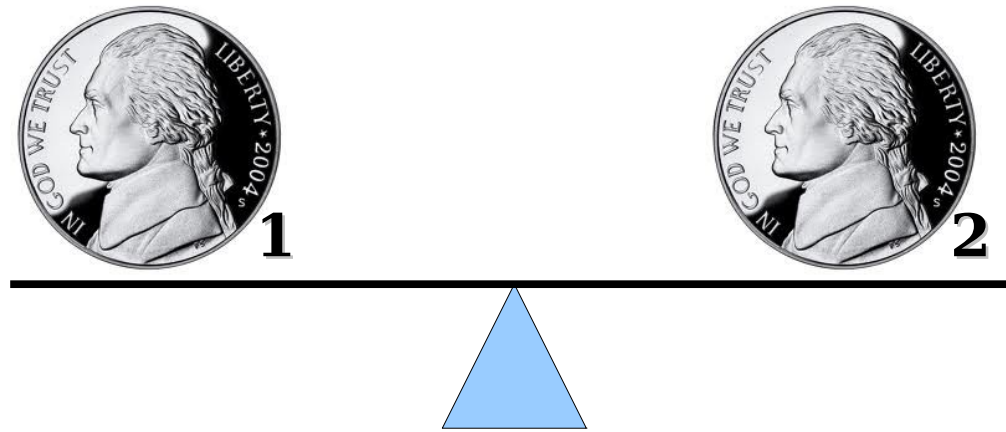
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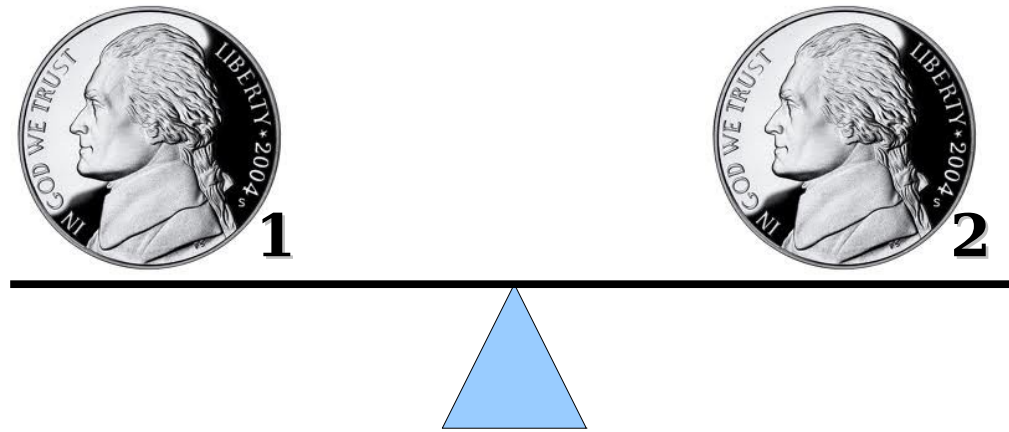
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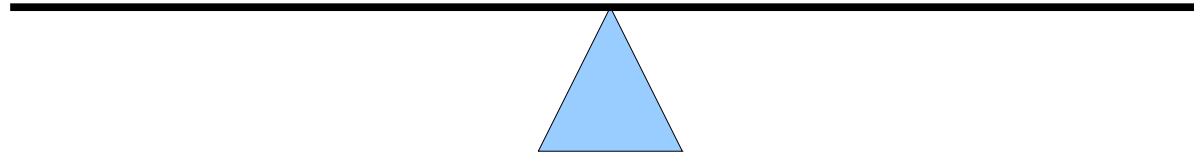
# Finding the Counterfeit Coin



# A Harder Problem

- You are given a set of *nine* seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only *two* weighings on the balance, find the counterfeit coin.

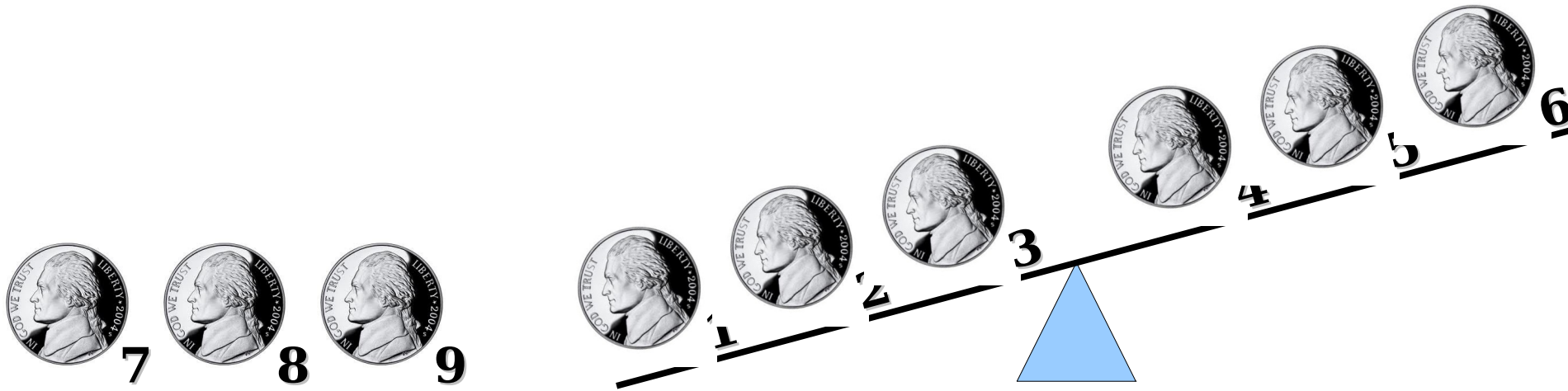
# Finding the Counterfeit Coin



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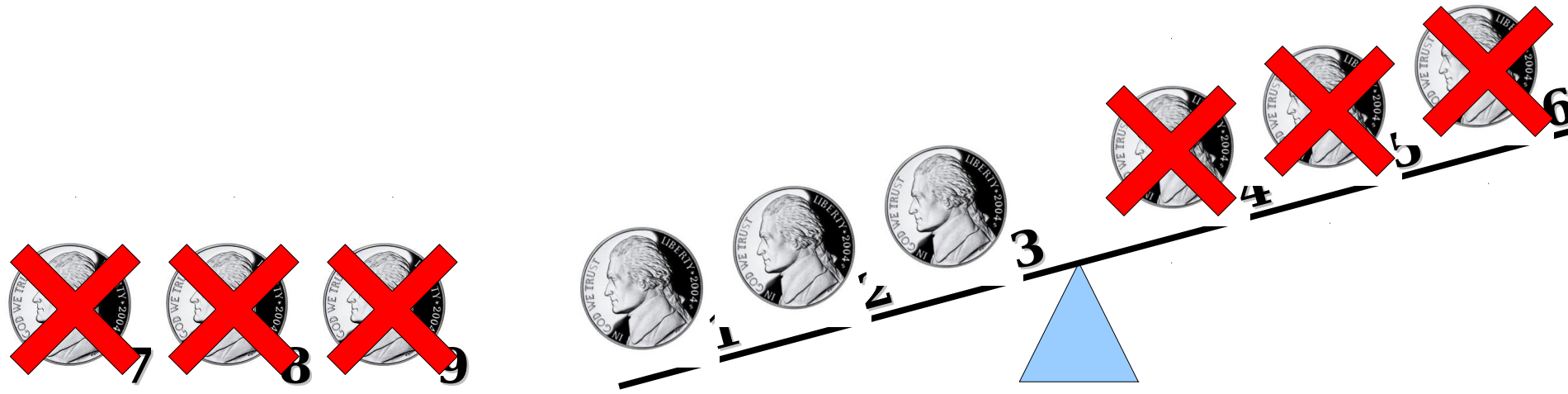


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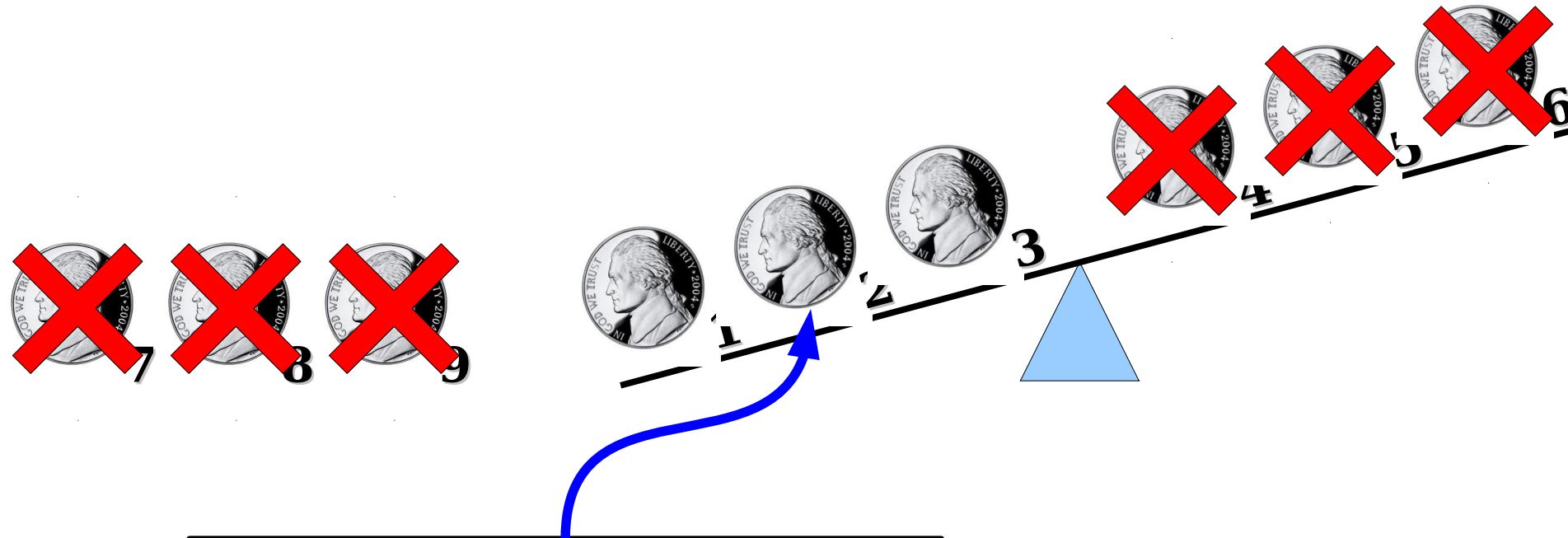




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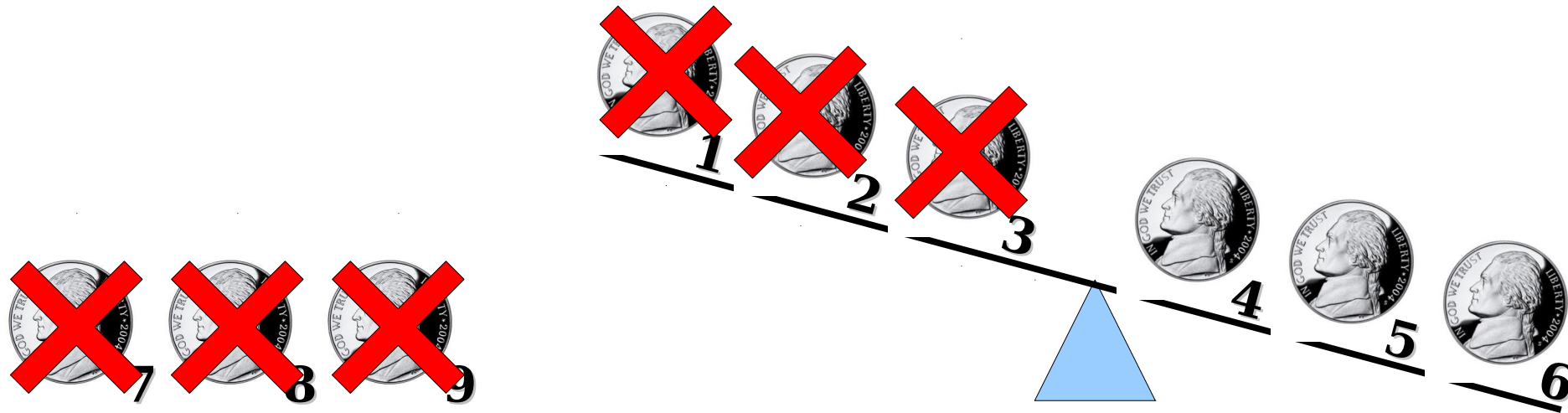


Now we have one weighing to find the counterfeit out of these three coins.

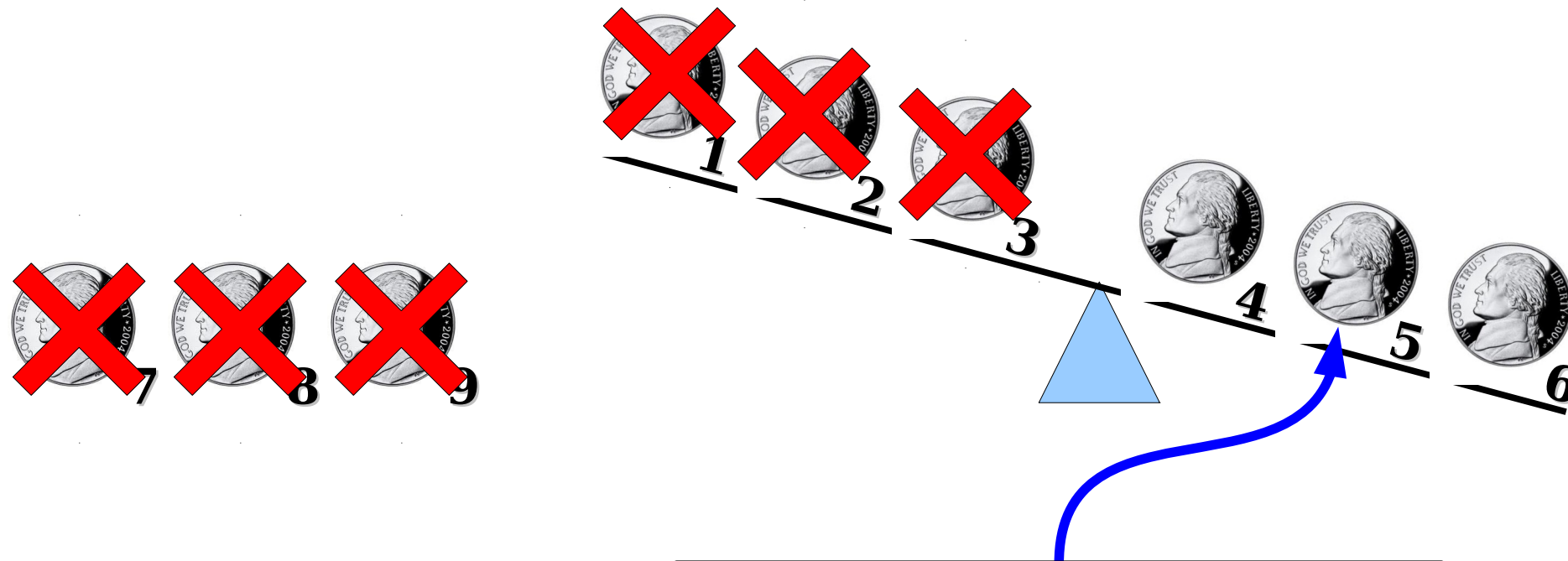
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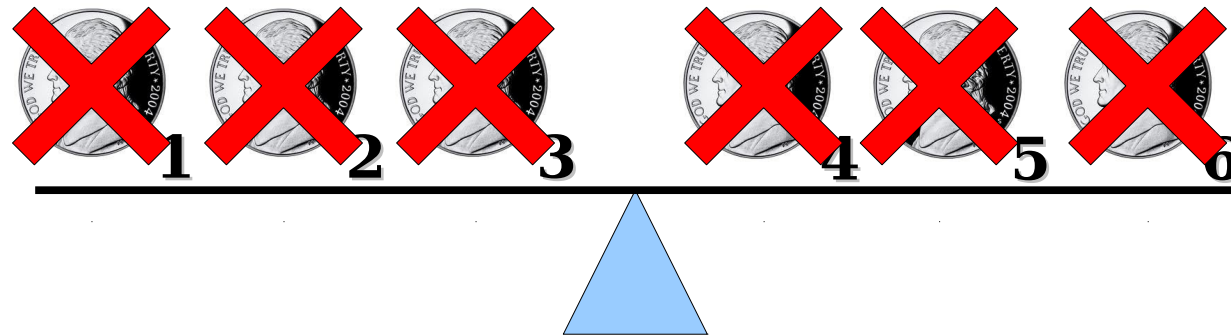


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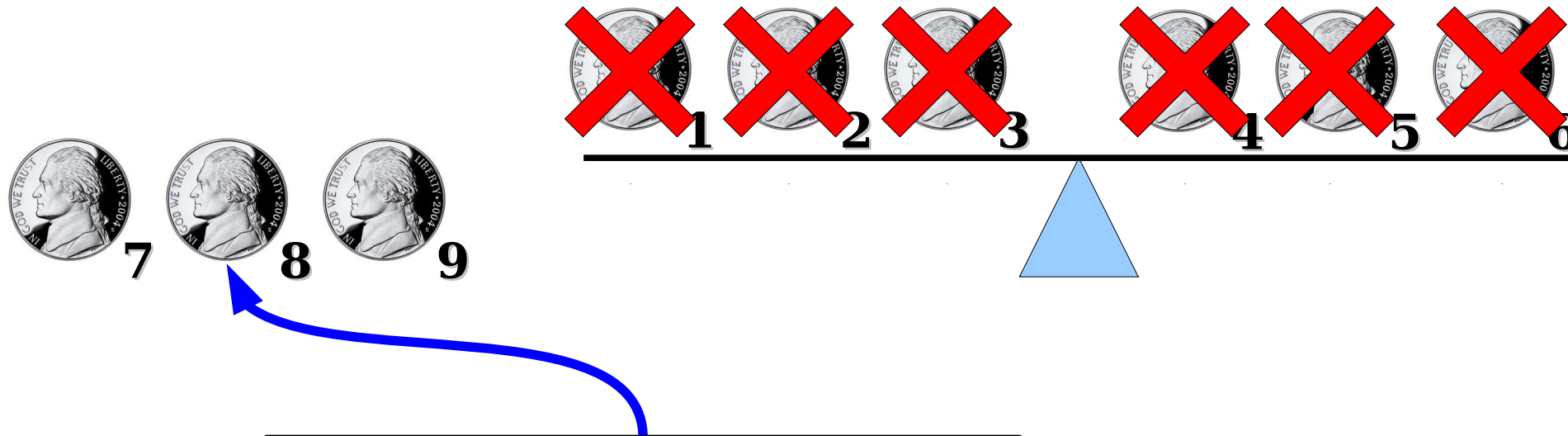
# Finding the Counterfeit Coin



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Now we have one weighing to find the counterfeit out of these three coins.



Can we generalize this?

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1 coin : ? weighings  
3 coins: 1 weighing  
9 coins: 2 weighings

**Quick check:** What is the pattern? The number of weighings we need to find a counterfeit in  $3^n$  coins is...  
Go to  
[PollEv.com/cs103spr25](https://PollEv.com/cs103spr25)

# A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

***Theorem:*** If exactly one coin in a group of  $3^n$  coins is heavier than the rest, that coin can be found using only  $n$  weighings on a balance.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers  $n$ , then tell them we're going to prove it by induction.

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Here, we state what  $P(0)$  actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

**“For all  $k \in \mathbb{N}$ , if  $P(k)$  is true, then  $P(k+1)$  is true.”**

To do this, we'll choose an arbitrary  $k$ , assume that  $P(k)$  is true, then try to prove  $P(k+1)$ .

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Here, we use our **inductive hypothesis** (the assumption that  $P(k)$  is true) to solve this simpler version of the overall problem.

If exactly one coin in a group of  $3^n$  coins is heavier than the rest, that coin can be found using only  $n$  weighings on a balance. We'll use induction to prove this theorem. The base case is  $n=0$ . As our base case, we have a set of  $3^0=1$  coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

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For the induction step, we can assume that  $P(k)$  is true for some  $k \in \mathbb{N}$ , and we want to prove that  $P(k+1)$  is true.

Suppose we have a group of  $3^{k+1}$  coins. We can divide these coins into three groups of  $3^k$  coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin. Otherwise, the heavier coin must be in the group we didn't put on the scale. Therefore, with one weighing, we can find a group of  $3^k$  coins containing the heavy coin. We can then use  $k$  more weighings to find the heavy coin in that group.

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In a proof by induction, we need to prove that

- ✓  $P(0)$  is true
- ✓ For all  $k \in \mathbb{N}$ , if  $P(k)$  is true, then  $P(k+1)$  is true.

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# Some Fun Problems

- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get  $k$  weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have  $k$  weighings?
  - Can you find the counterfeit out of a group of more than  $3^k$  coins with  $k$  weighings?
  - Can you find the counterfeit out of any group of at most  $3^k$  coins with  $k$  weighings?

How Not To Induct

Something's Wrong...

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We need to show that  $P(k + 1)$  holds, meaning that the sum of the first  $k + 1$  powers of two is  $2^{k+1}$ . To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k + 2^k \quad (\text{via (1)}) \\ &= 2(2^k) \\ &= 2^{k+1}. \end{aligned}$$

Therefore,  $P(k + 1)$  is true, completing the induction.

**Quick check:** Where is the error?  
Go to  
[PollEv.com/cs103spr25](http://PollEv.com/cs103spr25)

# Something's Wrong...

**Theorem:** The sum of the first  $n$  powers of two is  $2^n$ .

**Proof:** Let  $P(n)$  be the statement “the sum of the first  $n$  powers of two is  $2^n$ .” We will prove, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ , from which the theorem follows.

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Where did we  
prove the base  
case?

Therefore,  $P(k + 1)$  is true, completing the induction. ■

When writing a proof by induction,  
***make sure to prove the base case!***  
Otherwise, your proof is incomplete!

Why did this work?

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You can prove **anything** from a faulty assumption. This is called the **principle of explosion**.

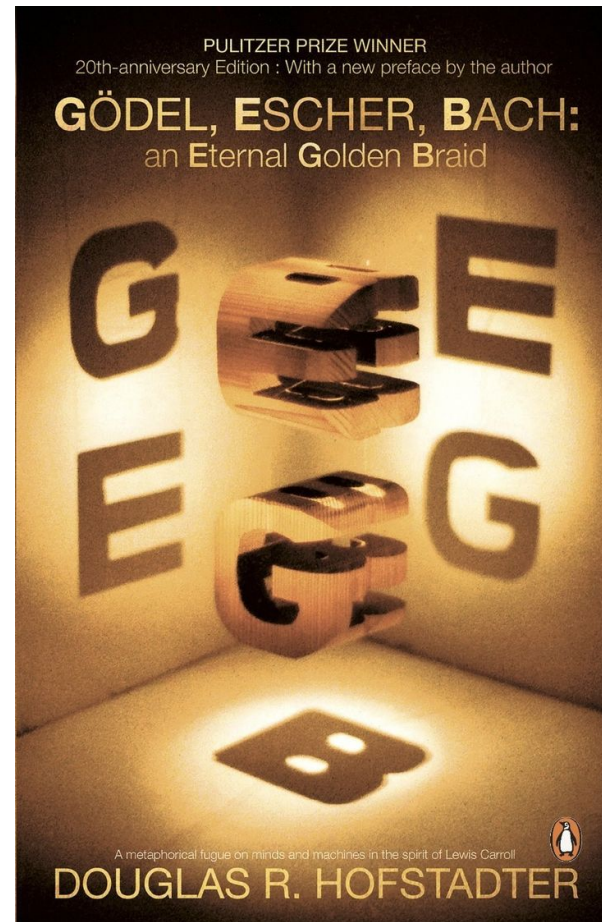
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# The MU Puzzle



# *Gödel, Escher Bach: An Eternal Golden Braid*

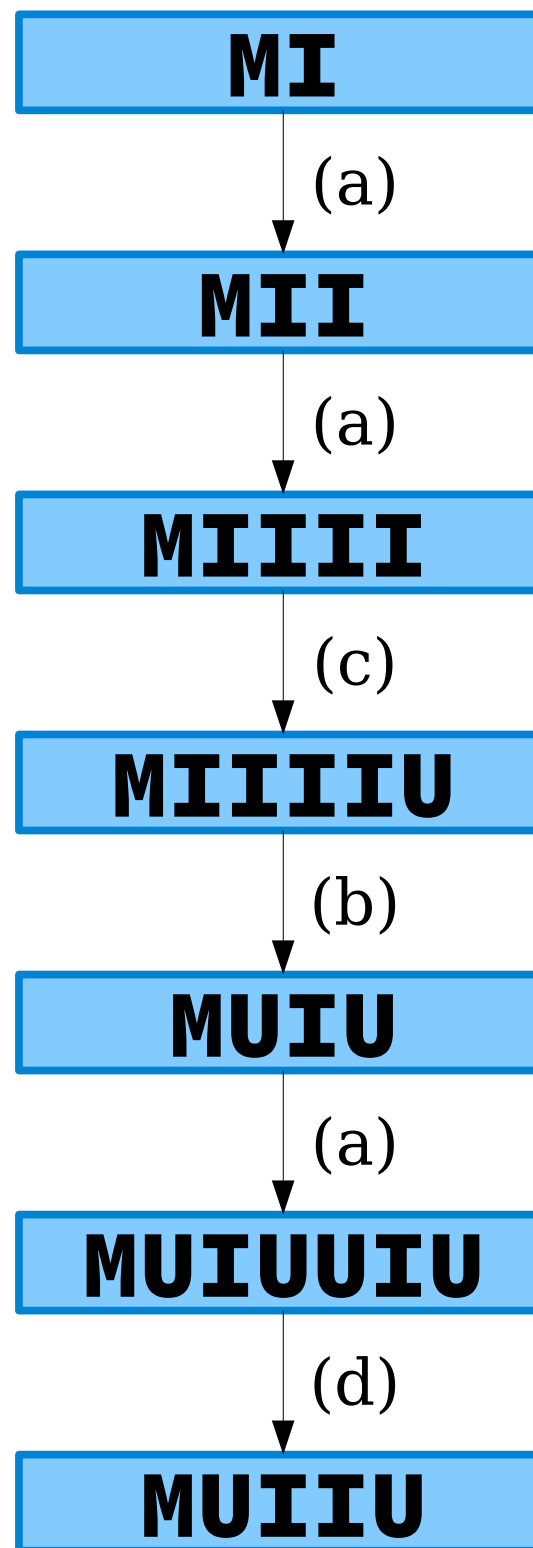
- Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.
- It's a great read after you've finished CS103 – you'll see so many of the ideas we'll cover presented in a totally different way!



# The MU Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **M**: for example, **MIU** becomes **MIUIU**, or **MI** becomes **MII**.
  - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**.
  - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**.
  - Remove any **UU**: **MUUU** becomes **MU**.
- **Question**: How do you transform **MI** to **MU**?

- (a) Double the string after an **M**.  
(b) Replace **III** with **U**.  
(c) Append **U**, if the string ends in **I**.  
(d) Delete **UU** from the string.



# Try It!

Starting with **MI**, apply these operations to make **MU**:

- (a) Double the string after an **M**.
- (b) Replace **III** with **U**.
- (c) Append **U**, if the string ends in **I**.
- (d) Delete **UU** from the string.

Not a single person in this room  
was able to solve this puzzle.

Are we even sure that there is a solution?

# Counting **I**'s



# The Key Insight

- Initially, the number of **I**'s is *not* a multiple of three.
- To make **MU**, the number of **I**'s must end up as a multiple of three.
- Can we *ever* make the number of **I**'s a multiple of three?

***Lemma 1:*** If  $n$  is an integer that is not a multiple of three, then  $n - 3$  is not a multiple of three.

***Lemma 2:*** If  $n$  is an integer that is not a multiple of three, then  $2n$  is not a multiple of three.



**Lemma 1:** If  $n$  is an integer that is not a multiple of three, then  $n - 3$  is not a multiple of three.

**Proof:** By contrapositive; we'll prove that if  $n - 3$  is a multiple of three, then  $n$  is also a multiple of three. Because  $n - 3$  is a multiple of three, we can write  $n - 3 = 3k$  for some integer  $k$ . Then  $n = 3(k+1)$ , so  $n$  is also a multiple of three, as required. ■

**Lemma 2:** If  $n$  is an integer that is not a multiple of three, then  $2n$  is not a multiple of three.

**Proof:** Let  $n$  be a number that isn't a multiple of three. If  $n$  is congruent to one modulo three, then  $n = 3k + 1$  for some integer  $k$ . This means  $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$ , so  $2n$  is not a multiple of three. Otherwise,  $n$  must be congruent to two modulo three, so  $n = 3k + 2$  for some integer  $k$ . Then  $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$ , and so  $2n$  is not a multiple of three. ■

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**Theorem:** The **MU** puzzle has no solution.

**Proof:** Assume for the sake of contradiction that the **MU** puzzle has a solution and that we can convert **MI** to **MU**. This would mean that at the very end, the number of **I**'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of **I**'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the **MU** puzzle has no solution. ■

# Algorithms and Loop Invariants

- The proof we just made had the form
  - “If  $P$  is true before we perform an action, it is true after we perform an action.”
- We could therefore conclude that after any series of actions of any length, if  $P$  was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

# Next Time

- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.